Arithmetic & Geometric Sequences and Series - ACTIVITES

Learning Objectives

LO1: Students to understand the meaning of 'geometric series'

LO2: Students to learn how to calculate a geometric series and the constant ratio

LO3: Students to learn the meaning of 'compound interest'

LO4: Students to learn how to independently calculate compound interest.

ACTIVITY ONE

Background and Worked Example

One of the applications of geometric series is the calculation of compound interest. Here the sum on which interest is paid includes the interest that has been earned in previous years.

For example, if £100 is invested at 5% per annum compound interest, then after 1 year the interest earned is £5 (100 x 0.05) and the capital invested for the second year is £105. The interest earned in the second year is then £5.25 (£105 x 0.05) and this capital amount is carried forward to year 3.

Presenting this in tabular form:

Beginning of year	**1**	**2**	**3**	**4**	**5**
Capital	100	105	110.25	115.7625	121.550625
Interest during the year	100 x 0.05 = 5	105 x 0.05 = 5.25	110.25 x 0.05 = 5.5125	115.7625 x 0.05 = 5.788125	

Reproducing the table in algebraic form with 'a' representing capital and 'i' representing the rate of interest (in decimal form):

Beginning of year	**1**	**2**	**3**	**4**	**5**
Capital	a	$a + ai = a(1+i)$	$a(1+i) + a(1+i)i = a(1+i)^2$	$a(1+i)^2 + a(1+i)^2 i = a(1+i)^2(1+i) = a(1+i)^3$	$a(1+i)^3 + a(1+i)^3 i = a(1+i)^3(1+i) =$

					$a(1+i)^4$
Interest during the year	ai	$a(1+i)i$	$a(1+i)^2i$	$a(1+i)^3i$	

Therefore the value in year n is $a(1+i)^{n-1}$. Hence the example above with a = 100, i = 0.05, the value at the beginning of year 5 is:

$100 \times (1 + 0.05)^4 = 121.550625$

The sum of a geometric series is obtained easily by considering the series in its algebraic form (see table above):

$a, ar, ar^2, ar^3, \ldots, ar^{n-1}$

The sum of the first n terms is:

$S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}$

Multiplying both sides by r:

$rS_n = ar + ar^2 + ar^3 + ar^4 + \ldots + ar^n$

(since $ar^{n-1} \times r = ar^{n-1+1} = ar^n$)

Subtracting rS_n from S_n gives:
$S_n - rS_n = a - ar^n$

Hence:
$S_n = (a - ar^n)/(1 - r) = a(1 - r^n)/(1 - r)$

Example 1:

Find the sum of the first 10 terms of the series

8, 4, 2, 1,…..

This is a geometric series since each term is obtained by multiplying the previous term by 0.5. Applying the formula above, the sum of the first 10 terms is:

$S_{10} = 8(1 – 0.5^{10})/(1 – 0.5) = 16 \times (1 – 0.00098) = 15.98$

Note the calculation of r^n yields a very small value of 0.00098 and as 'n' gets larger this value can only get smaller, e.g.

$0.5^5 = 0.03125$, $0.5^{15} = 0.000031$

Hence if the series has an infinite number of terms, r^n will be so small that in practise it will be zero. The formula for the sum of an infinite geometric progression is then:

$S_n = a(1 – 0)/(1 – r) = a/(1- r)$

TASK ONE

A financial analyst is analysing the prospects of a certain company. The company pays an annual dividend on its stock. A dividend of £5 has just been paid and the analyst estimates that the dividends will grow by 20% per year for the next five years, followed by annual growth of 10% per year for 5 years.

(a) Complete the following table:

Year	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Dividend										

(b) Then calculate the total dividend that will be paid for the next ten years.

TASK TWO

A stock begins to pay dividends with the first dividend, one year from now, expected to be £10. Each year the dividend is 10% larger than the previous year's dividend. In what year is the dividend paid larger than £100?

ACTIVITY TWO (Arithmetic Series)

TASK ONE

An economist believes that the size of a regional economy (as measured by GDP) can be accurately measured using an arithmetic progression.
He discovers that GDP over the past 5 years is as follows:

Year 1: $ 250 million
Year 2: $ 267.5 million
Year 3: $ 285 million
Year 4: $ 302.5 million
Year 5: $ 320 million

Assuming that the arithmetic progression is correct and a robust guide to the future, calculate the size of the regional economy in year 17 only, **using a formula**.

Arithmetic & Geometric Sequences and Series - ANSWERS

ACTIVITY ONE

TASK ONE

(a)

Year	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Dividend	£6.00	£7.20	£8.64	£10.37	£12.44	£13.69	£15.05	£16.56	£18.22	£20.04

(b) $S_n = a(1 – r^n)/(1 – r)$

The sum of the first 5 dividends is:

$S_5 = 6 \times (1 - 1.2^5)/(1 - 1.2) = 6 \times (1 - 2.488)/(-0.2) =$ £44.65

The sum from year 6 to 10 is:

$S_{6\text{-}10} = 13.69 \times (1 - 1.1\text{^}^5)/(1 - 1.1) = 13.69 \times (1 - 1.61)/(-0.1) =$ £83.55

In total: 44.65 + 83.55 = £128.2.

TASK TWO

In order to answer this question we need to recall the expression for the value of a geometric series in a given period:

$T_n = ar^{n-1}$

Hence we need to find the 'n' that gives a T_n of £100 given an 'a' of £10 and an 'r' of 1.1. Hence:

£100 = £10 x 1.1^{n-1}

£100/£10 = £10 = 1.1^{n-1}

In order to solve this problem we need to refer back to Guide One and the rules surrounding logs, namely:

$\log(a^n)=n \log(a)$

Therefore, taking the logs of both sides:

$\log(10) = (n-1)\log(1.1)$

$n-1 = \log(10)/\log(1.1) = 24.16$

Hence n = 24.16 + 1 = 25.16. Thus the dividend paid in year 26 will be greater than £100. In fact it will be £10 x 1.1^{26-1}= £108.35. The dividend in year 25 will then be £10 x 1.1^{25-1} = £98.50 (which is less than £100).

ACTIVITY TWO

TASK ONE

$U_n = U_1 + (n-1) \times d$

U_{17} = \$250 million + (16 x \$17.5 million)
= \$250 million + (16 x \$17.5 million)
= \$250 million + \$280 million
= \$530 million

Simple and Compound Interest - ACTIVITES

<u>Learning Objectives</u>

LO1: Students learn the meaning of key financial terms- bond, coupon, interest, simple and compound interest

LO2: Students learn how to calculate simple and compound interest

LO3: Students learn the distinction between simple and compound interest

LO4: Students learn the impact which the frequency of compounding has upon the size of the total return

Task One

A bond's coupon is the annual interest rate paid on the issuer's borrowed money, generally paid out semi-annually. The coupon is always tied to a bond's face or **par value**, and is quoted as a percentage of par. For instance, a bond with a par value of £1,000 and an annual interest rate of 4.5% has a coupon rate of 4.5% (£45).

Say you invest in a six-year bond paying 5% per year, annually. Assuming you hold the bond to maturity, you will receive 6 interest payments of £250 each, or a total of £1,500. Plus the par value of £1000.This coupon payment is simple interest.

You can do two things with that simple interest—spend it or reinvest it. Determine the total amount of money you will have after six years, assuming you can reinvest the interest at 5% per annum.

Task Two

So far, it has been assumed that compound interest is compounded once a year. In reality interest may be compounded several times a year, e.g. daily, weekly, monthly, quarterly, semi-annually or even continuously.

The value of an investment at the end of m compounding periods is:

$$P_t = P_0[1 + {}^{r}/_{m}]^{m \times t}$$

Where m is the number of compounding periods per year and t is the number of years.

Using this information, solve the following problem:

(a) £1,000 is invested for three years at 6% per annum compounded semi-annually. Calculate the total return after three years.

(b) What would the answer be if the interest was compounded annually?

(c) If the interest was compounded monthly is it true that the total amount after three years would be less than £1195.00?

(d) Using your answers to (a) – (c) what can you infer about the frequency of compounding and the size of the total return?

Task Three (Using the formula $P_t = P_0e^{rt}$)

It follows that when we compound interest continuously the value of the investment at the end of the period becomes $P_t = P_0e^{rt.}$

Worked example

A financial consultant advises you to invest £1,000 at 6% continuously compounded for three years. Find the total value of your investment.

$$P_t = P_0e^{rt} = £1,000 \text{ x } e^{(0.06 \text{ x } 3)} = £1,000 \text{ x } 2.7183^{0.18} = £1,197.22$$

In Excel to raise 'e' to the power of another number you use the "EXP" function.

(a) Use Excel to set up a table comparing the growth of £1 invested for 25 years at 20% assuming interest is compounded (i) annually; (ii) quarterly; (iii) monthly and (iv) continuously.
(b) Graph the outcome.
(c) What conclusion can be drawn regarding the frequency of compounding?

Simple and Compound Interest - ANSWERS

Task One

Time (years)	Cash Flow received	Interest Earned at 5%
1	£250	£69.07
2	£250	£53.88
3	£250	£39.41
4	£250	£25.63
5	£250	£12.50
6	£1,250	£0.00
Totals	**£2,500**	**£200.48**

When you reinvest a coupon, however, you allow the interest to earn interest. The precise term is "interest-on-interest," (i.e. compounding). Assuming you reinvest the interest at the same 5% rate and add this to the £1,500 you made, you would earn a cumulative total of £2,700.48, or an extra £200.48 (of course, if the interest rate at which you reinvest your coupons is higher or lower, your total returns will be more or less).

Task Two

(a) $P_3 = £1{,}000 \times (1 + {}^{0.06}/_{2})^{2 \times 3} = £1{,}000 \times (1.03)^6 = £1194.05$

(b)

Time (years)	Principal	Interest Earned	Total at End of Period
1	£1,000.00	£60.00	£1,060.00
2	£1,060.00	£63.60	£1,123.60
3	£1,123.60	£67.42	£1,191.02

Which is the same as $£1{,}000 \times (1+0.06)^3 = £1{,}191.02$.

(c) **FALSE** since we would get

Time (months)	Principal	Interest Earned	Total at End of Period
1	£1,000.00	£5.00	£1,005.00
2	£1,005.00	£5.03	£1,010.03
3	£1,010.03	£5.05	£1,015.08
..	..	..	..
..	..	..	..
35	£1,184.83	£5.92	£1,190.75
36	£1,190.75	£5.95	£1,196.70

or £1,000 x $(1 + {}^{0.06}/_{12})^{12 \times 3}$ = £1,000 x $(1.005)^{36}$ = £1,196.68 (note difference is due to rounding errors)

(d) Hence the greater the compounding frequency the greater the total return. Thus if we compound daily the total return would be:

£1,000 x $(1 + {}^{0.06}/_{365})^{365 \times 3}$ = £1,197.20.

To illustrate what happens as the compounding frequency is increased, consider the table below.

Compounding Frequency	Total at End of Period
1	£1,191.02
2	£1,194.05
4	£1,195.62
8	£1,196.41
12	£1,196.68
52	£1,197.09
365	£1,197.20
730	£1,197.21

1460	£1,197.21
5000	£1,197.22
10000	£1,197.22
50000	£1,197.22

As the value of m (the compounding frequency) increases the value of the investment becomes larger, but never exceeds £1.197.22

Note the final value is arrived at from:

$£1,000 \times (1 + {}^{0.06}/_{50000})^{50000 \times 3} = £1,197.22.$

Note: Higher ability students might want to consider the following explanation:
By allowing m to approach infinity interest is being added to the investment more and more frequently and can be regarded as being added continuously, such that:

$$\lim_{m \to \infty} £1000x\left(1 + \frac{0.06}{m}\right)^{mx3} = £1{,}197.22$$

Here we applied this formula:

$P_t = P_0[1 + {}^{r}/_{m}]^{m \times t}$

with P_0 set at £1,000, 'r' set at 0.06 and 't' set at 3 and varying m. If we now set P_0 at £1, 'r' at 100% (i.e. 1) and 't' set at 1 year we arrive at the following answer for 'm' set at 50,000:

$P_t = £1x\ [1 + {}^{1}/_{50,000}]^{50,000} = 2.7183$

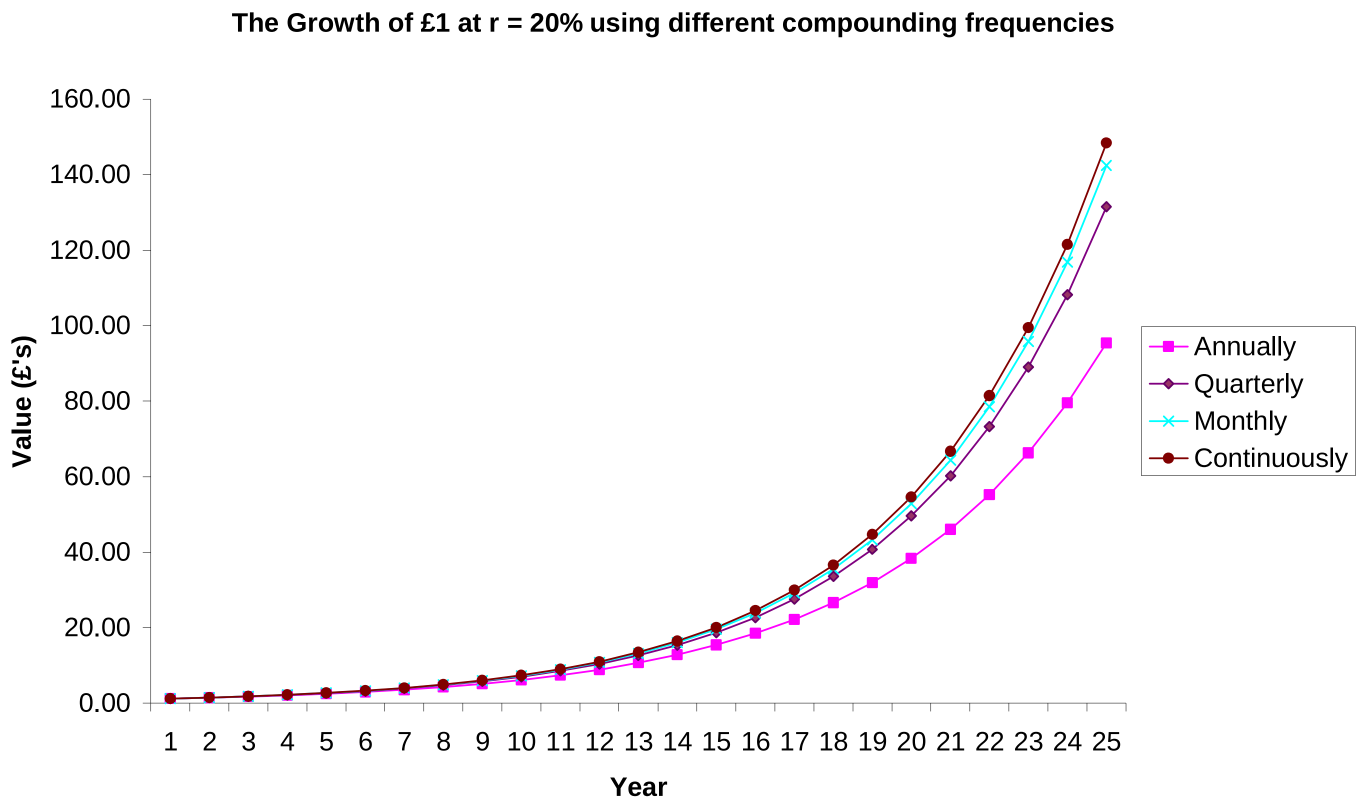
Thus we can say that:

$$\lim_{m \to \infty}\left(1 + \frac{1}{m}\right)^{m} = 2.7183$$

You may recognise this number, 2.7183, the natural logarithm, e. Note log to the base e of 2.7183 (denoted $\log_e(2.7183) = 1$).

Task Three

(a) and (b)



(c) We can observe from the chart that the continuous method of compounding leads to the greatest cumulative total after 20 years, while the annual method gives the smallest sum. Furthermore, the longer the investment is left on deposit the wider the differences when compounded by the different methods.

Investment Appraisal - ACTIVITIES

ACTIVITY ONE

<u>Learning Objectives</u>

LO1: Students to learn how to calculate present values

LO2: Students learn how to apply their understanding of present values to solve annuity problems

LO3: Students learn how to use formulae to solve financial problems

LO4: Students learn how to use Excel to answer financial questions

TASK ONE (Present Values and Annuities)

Suppose you win a £1m lottery prize and are offered the choice between taking the whole £1m now or £50,000 per year for 25 years. Which would you choose?

TASK TWO

Follow the worked example below and then attempt the task below (Se TASK)

Worked Example

The financial advisers Alexander Forbes quote annuity rates on there website:
http://www.annuity-bureau.co.uk/Annuity+Rates/Current+annuity+rates/

You can either choose an annuity with a fixed return or an annuity that increases in line with the RPI. A selection of quotes is presented below:

Escalation: Level

Last updated: 20 November 2006

Provider	Rank	Income	Provider	Rank	Income
Male 55, Single Life			Female 55, Single Life		
Scottish Equitable	1	£5,718.00	Norwich Union	1	£5,580.96
Canada Life	2	£5,713.92	Canada Life	2	£5,495.04
Prudential	5	£5,598.84	Prudential	5	£5,307.24
Friends Provident	7	£5,230.20	AXA Sun Life	7	£4,896.00

Source: http://www.annuity-bureau.co.uk/AF_CMS/Resources/Annuity%20bureau/rates_level.html (accessed 13/11/2006)

The derivation of the value of the figure of £5,718 from Scottish Equitable can be illustrated using the concept of Present Value. The income values given in the table quote how much £100,000 will purchase per annum for the rest of your life. Recall that to find the present value of an annuity we need to know the values for r and t, where t in this context will represent life expectancy.

The national statistics office (http://www.statistics.gov.uk/) maintains a database of life expectancy according to age and gender. These can be found by searching for "Interim life tables" at the above website. The interim life table indicates that the average life expectancy of a male aged 55 living in the United Kingdom would be 24.67 years (based on data for the years 2003-2005). For a woman it would be 28.04 years. Note this refers to the average number of years an individual will survive after the age of 55 years.

Source: http://www.statistics.gov.uk/downloads/theme_population/Interim_Life/ILTUK0305.xls

If interest rates were 5% then the PV of £1 received for the next 24.67 years would be:

$$\text{PV of annuity} = C \times \left[\frac{1}{r} - \frac{1}{r(1+r)^t}\right] = £1 \times \left[\frac{1}{0.05} - \frac{1}{0.05(1+0.05)^{24.67}}\right] = £13.9981$$

So if £100,000 was available to buy an annuity the annuity would be quoted as:

£100,000/£13.9981 = £7143.84

Note above the annuity from Scottish Life is quoted as £5,718. Using "Goal Seek in Excel" we can find the interest rate used by Scottish Life in their calculations.

C=	1
r=	2.88%
T=	24.67
PV=	17.4886
Annuity Rate=	£5,718.00

Or,

$$\text{PV of annuity} = C \times \left[\frac{1}{r} - \frac{1}{r(1+r)^t}\right] = £1 \times \left[\frac{1}{0.0288} - \frac{1}{0.0288(1+0.0288)^{24.67}}\right] = £17.4886$$

Hence we find that they used the rate of 2.88% (assuming the same life expectancy inputs).

TASK

Given the life expectancy values above determine the discount rates used by Canada Life in valuing their annuities.

ACTIVITY TWO

Learning Objectives

LO1: Students to learn how to calculate net present values and apply the IRR rule

LO2: Students to learn how to make independent evaluations of investment projects using NPV and IRR methodology

Task One (NPV)

Consider three alternative projects, A, B and C. They all cost £1,000,000 to set up but project's A and C return £800,000 per year for two years starting one year from set up. Projects B also returns £800,000 per year for two years, but the cash flows begin two years after set up. Whilst project C costs £1,000,000 to set up it initially requires £500,000 and £500,000 at termination (a clean-up cost for example).

If the interest rate is 20% which is the better project?

Task Two (IRR)

The Internal rate of return of a project can be defined as the rate of discount which, when applied to the projects cash flows, produces a zero NPV. That is, the IRR decision rule is then:

“invest in any project which has an IRR greater than or equal to some predetermined cost of capital”.

Consider a project that requires £4,000 investment and generates £2,000 and £4,000 in cash flows for two years, respectively. What is the IRR on this investment?

Investment Appraisal - ANSWERS

ACTIVITY ONE

Task One

If the interest rate were zero, the 25 payments of £50,000 would be chosen as this amounts to 25 x £50,000 = £1,250,000. However, interest rates are generally not zero and the present value of the £50,000 received in 10, 15 and 25 years time will be greatly reduced.

In order to determine the present value of the cash flows an appropriate interest rate needs to be determined. Lets assume an interest rate of 5%.

A regular payment over a fixed period of time is referred to as an annuity.

$$\text{PV of annuity} = C \times \left[\frac{1}{r} - \frac{1}{r(1+r)^t} \right]$$

Present value of the regular cash flow is £704,700. Hence the lottery winner should accept the £1m now.

TASK TWO

	Male	Female
C=	1	1
r=	2.87%	3.26%
T=	24.67	28.04
PV=	17.5011	18.1982
Annuity Rate=	£5,713.92	£5,495.04

ACTIVITY TWO

Task One

Note that the net cash flow for all three projects, ignoring the time value of money, is -£1,000,000 + £1,600,000 = +£600,000.

However, when the time value of money is taken into account one project may be preferable to the others. Without doing any calculations can you determine the order of preference?

Consider A versus B. They both cost the same but B's cash flow returns occur later than A's. Hence A is preferable to B.

Consider A versus C. They both have the same time pattern and size of returns and both cost the same to set up. However the payout to establish C is split with some cashflow up front and some at the end. Hence C is preferable to A.

Hence the rank is C, A, B.

However in many cases, the method of comparison is more complicated. In such cases, NPV analysis must be applied:

Project A

interest rate	**20%**	<<= you can change this			
Year	**0**	**1**	**2**	**3**	**4**
Cash Flow	-$1,000,000	$800,000	$800,000	$0	$0
discount factor	1.000	0.833	0.694	0.579	0.482
PV	-$1,000,000.00	$666,666.67	$555,555.56	$0.00	$0.00
NPV=	**$222,222.22**	Rank=	2		

Project B

interest rate	**20%**				
Year	**0**	**1**	**2**	**3**	**4**
Cash Flow	-$1,000,000	$0	$0	$800,000	$800,000
discount factor	1.000	0.833	0.694	0.579	0.482
PV	-$1,000,000.00	$0.00	$0.00	$462,962.96	$385,802.47
NPV=	**-$151,234.57**	Rank=	3		

Project C

interest rate	**20%**				
Year	**0**	**1**	**2**	**3**	**4**
Cash Flow	-$500,000	$800,000	$300,000	$0	$0
discount factor	1.000	0.833	0.694	0.579	0.482
PV	-$500,000.00	$666,666.67	$208,333.33	$0.00	$0.00
NPV=	**$375,000.00**	Rank=	1		

This is consistent with our previous intuitive analysis. These calculations are available at www.lawseconomics.co.uk/metal/npvandr.xls You will notice that if you lower the interest rate in cell B1 that the NPV rises and if you raise the interest rate the NPV falls.

Hence r↑, NPV↓ and r↓, NPV↑

Task Two

In order to verify the figure of 28.08% from above we can either find it by hand, using one of two ways, or use “Goal Seek” or the IRR function in Excel.

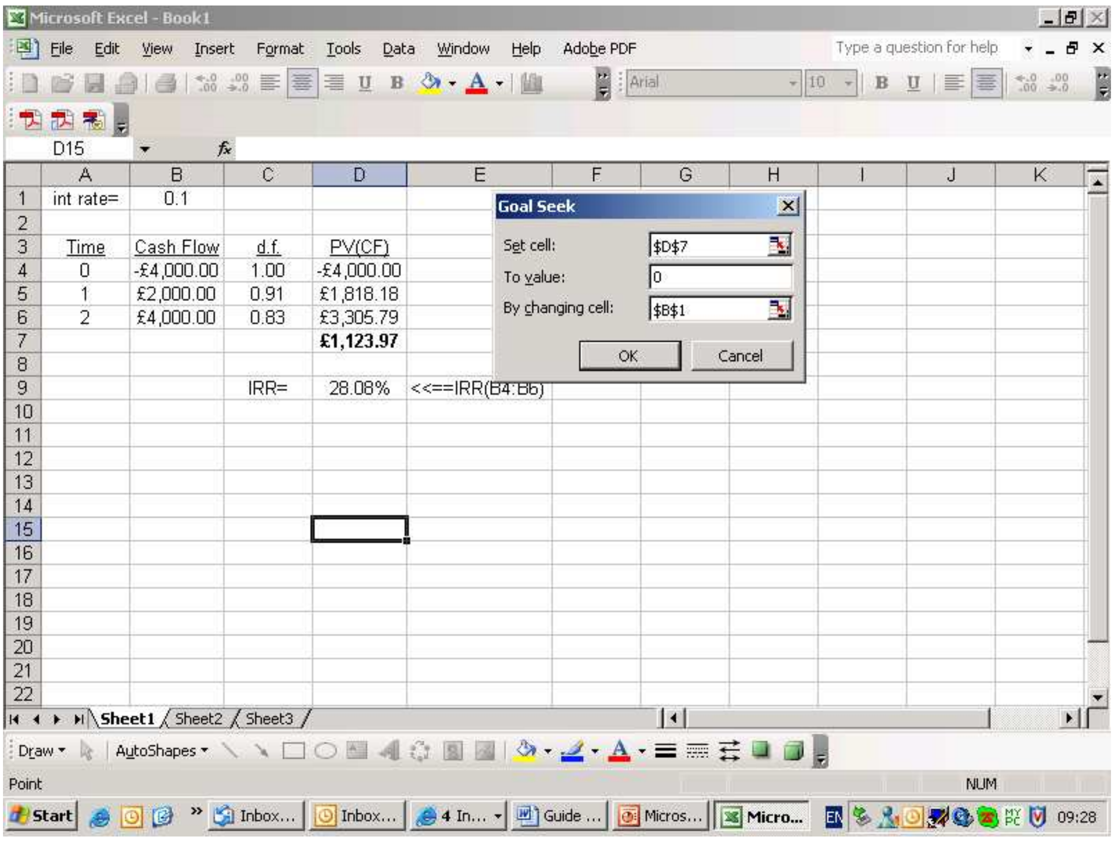
First of all using the IRR function. Note here we simply select the function and highlight the cash flows. We do not highlight the Present Values of the cash flows.

int rate= 0.1

Time	Cash Flow	d.f.	PV(CF)
0	- £4,000.00	1.00	- £4,000.00
1	£2,000.00	0.91	£1,818.18
2	£4,000.00	0.83	£3,305.79
			£1,123.97

IRR= 28.08%

Now using “Goal Seek” we arrive at the same answer.



int
rate= 0.2808

Time	Cash Flow	d.f.	PV(CF)
0	-£4,000.00	1.00	-£4,000.00
1	£2,000.00	0.78	£1,561.55
2	£4,000.00	0.61	£2,438.45
			£0.00

IRR= 28.08%

The graph below helps to illustrate the link between the interest rate and the NPV: